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The Green function in the theory of radiation and diffraction of regular water waves by a body*

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SUMMARY

This study is concerned with the Green function of the theory of potential flow about a body in regular (time-harmonic) water waves in deep water, that is with the linearized velocity potential of the flow due to a source of pulsating strength at a fixed position below the free surface (or a pulsating flux across the free surface) of a quiescent infinitely deep sea. An asymptotic expansion and a convergent ascending-series expansion for the Green function are obtained from two alternative complementary 'near-field' and 'far-field' single-integral representations in terms of the exponential integral. The asymptotic expansion and the ascending series allow efficient numerical evaluation of the Green function for large and small distances, respectively, from the mirror image of the singularity (submerged source or free-surface flux) with respect to the mean sea surface.

1. Introduction

A classical and important problem in free-surface hydrodynamics is that of linearized potential flow about a body in regular (time-harmonic) water waves. Particular problems of practical interest encompassed in this general potential-flow problem are the usual problems of wave radiation, in which a rigid body is forced to oscillate about a mean position in otherwise calm water, and of wave diffraction by a rigid body held fixed in a train of plane progressive waves. The problem of linearized motion of a freely floating rigid body in regular waves can be decomposed into such a wave-diffraction and six basic wave-radiation problems (corresponding to the six degrees of freedom of motion of an unrestrained rigid body), as is well known, and is explained in some detail in Wehausen [1] and Newman [2], for instance, where expressions for the wave force and moment and the coefficients of added mass and damping may also be found.

The present study is concerned with the Green function of the theory of potential flow about a body in regular water waves in deep water, that is with the linearized velocity potential of the flow due to a source of pulsating strength at a fixed position below the free surface (or a pulsating flux across the free surface) of a quiescent infinitely deep sea. This function has

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been studied extensively during the 1940s and early 1950s, notably by Kochin [3], Havelock [4, 5], Haskind [6, 7], John [8], Liu [9], Thorne [10], and MacCamy [11]. These studies are reviewed in Wehausen and Laitone [12], where several alternative integral representations of the Green function are listed. A convergent expansion of the Green function involving spherical harmonics is given in Ursell [13].

The advent of fast computers opening up the feasibility of numerical calculations for three-dimensional flows has caused a search of expressions for the Green function suited for efficient numerical evaluation. Thus, a modified form of the Haskind [7] expression for the Green function is given and used in Kim [14]. This modified Haskind expression was also used by Yeung [15], and was rederived by Hearn [16]. Recently, an integral representation for the Green function in terms of the exponential integral was obtained, independently and in different manners, by Guevel and Daubisse [17], Martin [18], and Noblesse [19]. Martin also gives asymptotic expansions of the Green function, and the study by Noblesse contains asymptotic expansions and convergent ascending series.

The plan and main results of the present study of the Green function are now presented. The basic potential-flow problem of the linearized theory of flow about a body in regular water waves is formulated in Section 2; following Lighthill [20], the problem is formulated as an initial-value problem of the form indicated by equation (2.2a). In Section 3, the classical double-integral representations (3.10a,b) of the Green function are derived, in a usual manner by using a double Fourier transform with respect to the horizontal coordinates x and y.

The field equation and boundary condition satisfied by the Green function are examined in detail in Section 4. Specifically, the Green function is shown to satisfy equations (4.3) or (4.4), depending on whether the pulsating singularity is fully submerged or on the mean sea surface, respectively. Equations (4.3) for a submerged source are well known. However, equations (4.4), corresponding to a flux across the mean sea surface, are proper in the limiting case when the singularity is exactly on the mean sea surface. These equations are important for the formulation of an integral equation for determining the velocity potential of flow about a body in regular waves, as is shown in [19] where a new integral equation indeed is obtained by using both equations (4.3) and (4.4).

In Section 5, three alternative and complementary single-integral representations of the Green function are obtained from the two alternative double-integral representations (3.10a,b). These three single-integral representations of the Green function, expressed in the form of equation (5.1), are: (i) the 'Haskind integral representation' (5.8c), which is essentially identical to the expression obtained by Haskind [7], (ii) the 'near-field integral representation' (5.11), which has also been obtained (independently and in a different manner) by Guevel and Daubisse [17] and Martin [18], and (iii) the 'far-field integral representation' (5.21), which does not appear to have been given previously.

The modified Haskind integral representation (5.8c) and the 'near-field representation' (5.11) are especially well suited for evaluating the function $g(h, v)$ for small values of v and h, respectively. Indeed, in the limiting cases $v = 0$ and $h = 0$ the integrals in these integral representations vanish, and the function $g(h, v)$ takes the simple forms given by equations (5.9) and (5.12), respectively. The near-field and far-field integral representations (5.11) and (5.21) are analogous to the single-integral representations of the Green function of ship wave-resistance theory given in Noblesse [21]. The integral representations (5.11) and (5.21) are used in Sections 6 and 7 for obtaining a convergent ascending series, useful for evaluating the function $g(h, v)$ for small and moderate values of $d = (h^2 + v^2)^{1/2}$, and an asymptotic expansion useful for large and moderate values of d.

In Section 6, two complementary asymptotic expansions of the function $g(h, v)$ for large and moderate values of d are obtained from the near-field and far-field integral representations (5.11) and (5.21). Comparison of these complementary expansions then yields the single expansion given by equations (6.17) and (6.17a,b). This asymptotic expansion is more general than the two asymptotic expansions for large values of h and v given in [18]. In Section 7, a convergent ascending-series expansion of the function $g(h, v)$ is obtained from the near-field integral representation (5.11). This ascending series, given by equations (7.7), (7.8) and (7.22), is useful for evaluating the function $g(h, v)$ for small and moderate values of d. The above-mentioned asymptotic expansion and ascending series are the two main new results of the present study.

These expansions are supplemented by two one-dimensional Taylor-series expansions useful for evaluating the function $g(h, v)$ in the vicinity of the axes $h = 0$ and $v = 0$. These series, obained from the near-field and the Haskind integral representations (5.11) and (5.8c), are given by equations (8.8) and (8.13) in Section 8. Finally, expressions for the gradient of the Green function are given in Section 9. In particular, the vertical derivative, G_z, of the Green function G can be directly expressed in terms of G, as is shown in expression (9.5). This expression, previously given in Martin [18], has been obtained here by following an idea used by Eggers [22] for the analogous problem of ship wave resistance.

2. The problem of potential flow about a body in regular water waves

The basic potential-flow problem of the linearized theory of flow about a body in regular water waves is briefly formulated in this section. A sea of infinite depth and lateral extent is assumed, and water is regarded as homogeneous, incompressible (with density ρ), and inviscid. The only body force considered is that due to a uniform gravitational field (with acceleration g). Surface tension and free-surface nonlinearities are neglected. The flow is irrotational and thus can be represented by a velocity potential Φ', which is a function of the Cartesian coordinates $\mathbf{X}(X, Y, Z)$ and of the time T, i.e. $\Phi'(\mathbf{X}, T)$. The mean (undisturbed) free surface of the sea is taken as the plane $Z = 0$, with the Z-axis positive upwards.

The linerized dynamic sea-surface boundary condition takes the well-known form

$$gE' + \Phi'_T + P'/\rho = 0 \qquad \text{on} \qquad Z = 0,$$

where $E'(X, Y, T)$ is the elevation of the free surface above or below its mean level $Z = 0$, $P'(X, Y, T)$ is the difference between the pressure at the free surface and the atmospheric pressure, and $\Phi'_T \equiv \partial\Phi'(X, Y, Z = 0, T)/\partial T$. For most problems of practical interest, the pressure at the sea surface is a constant equal to the atmospheric pressure, so that one then has $P' = 0$. In the presence of a fluid flux, $Q'(X, Y, T)$ say, across the sea surface, the linearized kinematic sea-surface boundary condition takes the form

$$\Phi'_Z = E'_T - Q' \qquad \text{on} \qquad Z = 0,$$

where $Q' < 0$ corresponds to fluid being sucked away across the free surface. While for all practical problems we have $Q' = 0$, it will be useful to allow a fluid flux across the free surface for determining the sea-surface condition satisfied by the Green function, as will be shown in Section 4. Elimination of the sea-surface elevation E' between the foregoing dynamic and kinematic sea-surface conditions then yields the sea-surface boundary condition

$$g\Phi'_Z + \Phi'_{TT} = -P'_T/\rho - gQ' \qquad \text{on} \qquad Z = 0, \tag{2.1}$$

which thus involves the velocity potential Φ' alone.

In the present study, we are interested in flows that are simple harmonic in time, say with radiant frequency ω (period $2\pi/\omega$). As is well known, and is discussed for instance in Stoker [23], such free-surface gravity flows are not completely (uniquely) determined unless one imposes a 'radiation condition' expressing that waves at a sufficient distance away from the disturbance (for instance, a body) which created them must be like 'outgoing' progressive waves, i.e. like progressive waves moving away from the waves source. A convenient alternative approach, employed for instance in Lighthill [20], to the use of such a 'radiation condition' of 'outgoing waves', consists in defining a time-harmonic flow as the limit – as the small positive auxiliary parameter ϵ vanishes – of a flow defined by a velocity potential of the form

$$\Phi'(\mathbf{X}, T) = \operatorname{Re} \Phi(\mathbf{X}) \exp\,[-i\omega(1+i\epsilon)T], \tag{2.2a}$$

where Re represents the real part. The sea-surface pressure and flux are similarly assumed to be of the form

$$P'(X, Y, T) = \operatorname{Re} P(X, Y) \exp\,[-i\omega(1+i\epsilon)T], \tag{2.2b}$$

$$Q'(X, Y, T) = \operatorname{Re} Q(X, Y) \exp\,[-i\omega(1+i\epsilon)T]. \tag{2.2c}$$

In this alternative approach, one then is faced with a traditional 'initial-value problem', with the obvious initial conditions $\Phi' = 0$ and $\Phi'_T = 0$ for $T = -\infty$. Use of expressions (2.2a, b, c) into equation (2.1) then yields the following sea-surface boundary condition:

$$g\Phi_Z - \omega^2(1+i\epsilon)^2\Phi = i\omega(1+i\epsilon)P/\rho - gQ \qquad \text{on} \qquad Z = 0, \tag{2.3}$$

for the 'spatial component' $\Phi(\mathbf{X})$ of the actual potential $\Phi'(\mathbf{X}, T)$.

It will be convenient to define adimensional variables in terms of $1/\omega$ as reference time and of some reference length L, from which the reference velocity ωL, potential ωL^2, and pressure $\rho\omega^2 L^2$ can be readily formed. We thus define the adimensional variables

$$t = \omega T, \qquad \mathbf{x} = \mathbf{X}/L, \qquad \phi = \Phi/\omega L^2, \qquad p = P/\rho\omega^2 L^2, \qquad q = Q/\omega L. \tag{2.4}$$

In terms of these adimensional variables, the sea-surface condition (2.3) can be shown to become

$$\phi_z - f(1+i\epsilon)^2\phi = if(1+i\epsilon)p - q \quad \text{on} \quad z = 0, \tag{2.5}$$

where f is the 'frequency parameter' defined as

$$f = \omega^2 L/g. \tag{2.5a}$$

The 'frequency parameter' f can obviously be made equal to unity by selecting the reference length L as g/ω^2. This choice of reference length essentially corresponds to taking the length of the water waves as reference length, since we have $g/\omega^2 = \lambda/2\pi$ – with λ the wavelength of plane progressive waves of frequency ω – from the 'dispersion relation' for water waves in deep water. In this choice of reference length, the size of the body causing the waves would however appear to vary with the frequency ω (the body becoming small at low frequency and large at high frequency). An alternative (possibly more convenient for practical purposes) choice is to take the reference length L as a length characterizing the size of the body, which would thus remain the same at all frequencies. The length of the waves, however, would then vary with the frequency (the waves being long at low frequency and short at high frequency).

The basic potential-flow problem of the linearized theory of flow about a body in regular water waves can now be briefly stated. As is well known, this problem consists in solving the Laplace equation

$$\nabla^2\phi = 0 \quad \text{in} \quad (d), \tag{2.6a}$$

subject to the boundary conditions specified below. The solution domain (d) in equation (2.6a) is the domain exterior to the body and bounded upwards by the mean sea surface, (σ) say, which consists in the whole plane $z = 0$ if the body is fully submerged or in the portion of the plane $z = 0$ exterior to the body in the case where the body pierces the free surface. On the mean sea surface (σ), the sea-surface boundary condition (2.5) must be satisfied:

$$\phi_z - f(1+i\epsilon)^2\phi = if(1+i\epsilon)p - q \quad \text{on} \quad (\sigma), \tag{2.6b}$$

where in fact we generally have $p = 0 = q$ for the problem of flow about a body. The potential ϕ must vanish at infinity; specifically, we have the condition

$$\phi = O(1/|\mathbf{x}|) \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{2.6c}$$

expressing that ϕ vanishes at least as fast as $1/|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$. Finally, on the body surface, (b) say, which actually consists only in the portion of the body surface located below the plane $z = 0$ if the body pierces the sea surface, the potential ϕ must satisfy the usual body boundary condition

$$\phi_n \text{ given on } (b), \tag{2.6d}$$

where $\phi_n \equiv \partial\phi/\partial n \equiv \nabla\phi \cdot \mathbf{n}$ is the derivative of ϕ in the direction of the unit normal $\mathbf{n}$ to (b).

The precise form taken by the expression for ϕ_n on (b) in particular problems, notably in the usual 'radiation' and 'diffraction' problems, may be found in various places in the literature, e.g. in Wehausen [1] and Newman [2].

3. Double-integral representations

A well-known technique for solving a potential-flow problem such as the one defined above by equations (2.6a, b, c, d) in the general case of an arbitrarily shaped body, consists in formulating an integral equation for the potential ϕ based on the use of a Green function satisfying all the boundary conditions of the problem except the 'body condition', which is to be satisfied by means of the integral equation. The Green function, $G(\mathbf{x};\boldsymbol{\xi},f,\epsilon)$ say, appropriate to the present problem then is the solution of the problem defined by the following equations:

$$\nabla^2 G = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) \quad \text{in} \quad z<0, \tag{3.1a}$$

$$G_z - f(1+i\epsilon)^2 G = 0 \quad \text{on} \quad z=0, \tag{3.1b}$$

$$G = O(1/r) \quad \text{as} \quad r\to\infty, \tag{3.1c}$$

where $\delta(\ )$ is the usual 'Dirac delta function', and $r\equiv|\mathbf{x}-\boldsymbol{\xi}|$ is the distance between the 'field point' $\mathbf{x}$ and the 'singular point' $\boldsymbol{\xi}$.

A particular solution of the Poisson equation (3.1a) is given by $4\pi G=-1/r$, as is well known and can readily be verified. The general solution of equation (3.1a) can thus be written as

$$4\pi G(\mathbf{x};\boldsymbol{\xi},f,\epsilon) = -1/r + H(\mathbf{x};\boldsymbol{\xi},f,\epsilon), \tag{3.2}$$

where the function H is regular harmonic in the lower half space $z<0$, and evidently is to be determined from the boundary conditions (3.1b, c). Indeed, use of expression (3.2) into equations (3.1a, b, c) yields

$$\nabla^2 H = 0 \quad \text{in} \quad z<0, \tag{3.3a}$$

$$H_z - f(1+i\epsilon)^2 H = [\partial_z - f(1+i\epsilon)^2](1/r) \quad \text{on} \quad z=0, \tag{3.3b}$$

$$H = O(1/r) \quad \text{as} \quad r\to\infty. \tag{3.3c}$$

The above problem can be solved by using a double Fourier transform with respect to the horizontal coordinates x and y. The double Fourier transform of the function $H(\mathbf{x};\boldsymbol{\xi},f,\epsilon)$ is denoted by $H^{**}(\alpha,\beta,z;\boldsymbol{\xi},f,\epsilon)$ and defined as

$$H^{**}(\alpha,\beta,z;\boldsymbol{\xi},f,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx e^{i(\alpha x+\beta y)} H(\mathbf{x};\boldsymbol{\xi},f,\epsilon). \tag{3.4}$$

The corresponding Fourier transform of the function $1/r$ is

$$(1/r)^{**} = (1/k)\exp\left[-k|z-\zeta| + i(\alpha\xi + \beta\eta)\right] \tag{3.5}$$

where $k \equiv (\alpha^2 + \beta^2)^{1/2}$ by definition, as may be verified. By taking the double Fourier transform with respect to x and y of equations (3.3a,b,c), we may then obtain the following 'Fourier-transformed problem' for the function $H^{**}(z;\alpha,\beta,\boldsymbol{\xi},f,\epsilon)$:

$$d^2H^{**}/dz^2 - k^2H^{**} = 0 \quad \text{in} \quad z<0, \tag{3.6a}$$

$$dH^{**}/dz - f(1+i\epsilon)^2H^{**} = -\left[1 + f(1+i\epsilon)^2/k\right] e^{k\zeta + i(\alpha\xi+\beta\eta)} \quad \text{on} \quad z = 0, \tag{3.6b}$$

$$H^{**} \to 0 \quad \text{as} \quad z \to -\infty. \tag{3.6c}$$

The general solution of the ordinary differential equation (3.6a) is $H^{**} = A\exp(kz) + B\exp(-kz)$, where A and B are arbitrary constants. The boundary condition (3.6c) shows that $B=0$, and the constant A then can be determined from the sea-surface condition (3.6b). We thus may obtain

$$H^{**} = -\frac{k+f(1+i\epsilon)^2}{k-f(1+i\epsilon)^2}\,\frac{1}{k}\,e^{k(z+\zeta)+i(\alpha\xi+\beta\eta)}, \tag{3.7}$$

which can be expressed in the equivalent forms

$$H^{**} = -\frac{1}{k}e^{k(z+\zeta)+i(\alpha\xi+\beta\eta)} - \frac{2f(1+i\epsilon)^2}{k-f(1+i\epsilon)^2}\,\frac{1}{k}\,e^{k(z+\zeta)+i(\alpha\xi+\beta\eta)}, \tag{3.7a}$$

$$H^{**} = \frac{1}{k}e^{k(z+\zeta)+i(\alpha\xi+\beta\eta)} - \frac{2}{k-f(1+i\epsilon)^2}\,e^{k(z+\zeta)+i(\alpha\xi+\beta\eta)}. \tag{3.7b}$$

It may be seen from equation (3.5) that the first term on the right side of equation (3.7b) is equal to the double Fourier transform $(1/r')^{**}$ of $1/r'$, where r' is defined as $r' \equiv (x'^2 + y'^2 + z'^2)^{1/2}$ with $x' \equiv x - \xi$, $y' \equiv y - \eta$, and $z' \equiv z + \zeta$. Thus, $\mathbf{x}'(x',y',z')$ is the vector joining the mirror image of the 'singularity' $\boldsymbol{\xi}$ with respect to the sea surface $z=0$ to the 'field point' $\mathbf{x}$, and r' is the distance between these two points.

The function $H(\mathbf{x};\boldsymbol{\xi},f,\epsilon)$ may now be obtained by taking the inverse double Fourier transform of the function $H^{**}(\alpha,\beta,z;\boldsymbol{\xi},f,\epsilon)$, namely

$$H(x,y,z;\boldsymbol{\xi},f,\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha\, e^{-i(\alpha x+\beta y)} H^{**}(\alpha,\beta,z;\boldsymbol{\xi},f,\epsilon). \tag{3.8}$$

By using expressions (3.7a,b) for H^{**} into equation (3.8), and by using the resulting expression for H into equation (3.2), we can then obtain the following alternative expressions for the Green function $G(\mathbf{x};\boldsymbol{\xi},f,\epsilon)$:

$$4\pi G = -\frac{1}{r} - \frac{1}{r'} - \frac{f(1+i\epsilon)^2}{\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{e^{kz' - i(\alpha x' + \beta y')}}{k[k - f(1+i\epsilon)^2]}, \tag{3.9a}$$

$$4\pi G = -\frac{1}{r} + \frac{1}{r'} - \frac{1}{\pi} \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{e^{kz' - i(\alpha x' + \beta y')}}{k - f(1+i\epsilon)^2}. \tag{3.9b}$$

The Green function $G(\mathbf{x}; \boldsymbol{\xi}, f, \epsilon)$ obviously is axisymmetric about the vertical axis $x = \xi, y = \eta$, so that we may take $y' \equiv y - \eta$ as zero and replace $x' \equiv x - \xi$ by $\rho \equiv (x'^2 + y'^2)^{1/2}$ in expressions (3.9a,b). Expression (3.9b) then becomes

$$4\pi G = -\frac{1}{r} + \frac{1}{r'} - \frac{2}{\pi} \int_{0}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha \frac{e^{z'(\alpha^2 + \beta^2)^{1/2} - i\rho\alpha}}{(\alpha^2 + \beta^2)^{1/2} - f(1+i\epsilon)^2}. \tag{3.10a}$$

A more usual alternative expression is that which can be obtained by performing the substitution $y' = 0$ and $x' = \rho$ in expression (3.9a), followed by a transformation from the Cartesian Fourier variables α and β to the polar variables $k \equiv (\alpha^2 + \beta^2)^{1/2}$ and θ, specifically by performing the change of variable $\alpha = k \cos\theta$ and $\beta = k \sin\theta$. The resulting classical expression is

$$4\pi G = -\frac{1}{r} - \frac{1}{r'} - f(1+i\epsilon)^2 \frac{2}{\pi} \int_{0}^{\pi} d\theta \int_{0}^{\infty} dk \frac{e^{(z' - i\rho\cos\theta)k}}{k - f(1+i\epsilon)^2}. \tag{3.10b}$$

Expressions (3.10a,b) show that $G(\mathbf{x}; \boldsymbol{\xi}, f, \epsilon)$ actually is a function of only three space variables, namely $\rho \equiv (x'^2 + y'^2)^{1/2}$, $z' \equiv z + \zeta$, and $(z - \zeta)^2$ which occurs in $r \equiv [\rho^2 + (z - \zeta)^2]^{1/2}$.

In Section 5, three alternative single-integral representations of the Green function will be obtained from the two alternative double-integral representations (3.10a,b). The single-integral representations will then be used to obtain asymptotic expansions, a convergent ascending-series expansion, and one-dimensional Taylor-series expansions in Sections 6, 7, and 8, respectively. However, before proceeding with the derivation of these expansions, we shall first examine the field equation and sea-surface boundary condition satisfied by the Green function, in the following section.

4. Field equation and boundary condition satisfied by the Green function

As is self-evident from equations (3.1a,b,c), the physical significance of the Green function $G(\mathbf{x}; \boldsymbol{\xi}, f, \epsilon)$ is that $\operatorname{Re} G(\mathbf{x}, \boldsymbol{\xi}, f, \epsilon) \exp(\epsilon - i)t$ is the linearized velocity potential, at the 'field point' $\mathbf{x}(x, y, z \leqslant 0)$ and at the time t, of the flow caused by a submerged pulsating source of strength $\operatorname{Re} \exp(\epsilon - i)\tau$, $-\infty \leqslant \tau \leqslant t$, located at point $\boldsymbol{\xi}(\xi, \eta, \zeta < 0)$. However, this well-known physical interpretation becomes ambiguous in the limiting case $\zeta = 0$, since the source then is obviously no longer fully submerged. A natural complementary interpretation for this limiting case is to assume that the outflow produced at point $(\xi, \eta, \zeta = 0)$ now stems from a flux $\operatorname{Re} q(x, y) \exp(\epsilon - i)t$, with $q(x, y) = \delta(x - \xi)\delta(y - \eta)$, across the mean sea surface $z = 0$. Equations (2.6a,b,c) then suggest that the 'limit Green function' $G_l(\mathbf{x}; \xi, \eta, f, \epsilon) \equiv G(\mathbf{x}; \xi, \eta, \zeta = 0, f, \epsilon)$ must satisfy the following equations:

$$\nabla^2 G_l = 0 \quad \text{in} \quad z < 0, \tag{4.1a}$$

$$G_{lz} - f(1+i\epsilon)^2 G_l = -\delta(x-\xi)\delta(y-\eta) \quad \text{on} \quad z = 0, \tag{4.1b}$$

$$G_l = O(1/r) \quad \text{as} \quad r \to \infty. \tag{4.1c}$$

A mathematical demonstration of the above physically motivated equations can readily be provided by verifying that the solution $G_l(\mathbf{x};\, \xi, \eta, f, \epsilon)$ of the problem defined by equations (4.1a,b,c) actually is identical to the 'limit Green function' obtained by replacing ζ by zero in the previously derived solution $G(\mathbf{x};\, \boldsymbol{\xi}, f, \epsilon)$ of the problem defined by equations (3.1a,b,c). Problem (4.1) may be solved in the same manner as was used previously for solving problem (3.3), namely by using a double Fourier transform with respect to the horizontal coordinates x and y. We may then obtain the 'Fourier-transformed' problem:

$$d^2 G_l^{**}/dz^2 - k^2 G_l^{**} = 0 \quad \text{in} \quad z < 0,$$

$$dG_l^{**}/dz - f(1+i\epsilon)^2 G_l^{**} = -\exp\,[i(\alpha\xi + \beta\eta)]/2\pi \quad \text{on} \quad z = 0,$$

$$G_l^{**} \to 0 \quad \text{as} \quad z \to -\infty,$$

where G_l^{**} is the double-Fourier transform of G_l, as is defined by formula (3.4). The solution of the above problem is given by

$$G_l^{**} = -\exp\,[kz + i(\alpha\xi + \beta\eta)]/2\pi[k - f(1+i\epsilon)^2].$$

By taking the inverse double Fourier transform, as is given by formula (3.8), we can finally obtain

$$G_l(\mathbf{x};\, \xi, \eta, f, \epsilon) = \frac{-1}{4\pi^2}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\alpha\, \frac{e^{kz - i(\alpha x' + \beta y')}}{k - f(1+i\epsilon)^2}, \tag{4.2}$$

which can readily be verified to be identical to the expression obtained by replacing ζ by zero in formula (3.9b).

Conversely, it may be shown that the 'limit Green function' G_l given by expression (4.2) does in fact satisfy equations (4.1a,b,c). Verification of equations (4.1a) and (4.1c) can easily be checked. As for the sea-surface condition (4.1b), we have

$$G_{lz} - f(1+i\epsilon)^2 G_l = -\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x'}\, d\alpha\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta y'}\, d\beta\right) \quad \text{on} \quad z = 0,$$

from which we may obtain

$$G_{lz} - f(1+i\epsilon)^2 G_l = -\delta(x')\delta(y') = -\delta(x-\xi)\delta(y-\eta) \quad \text{on} \quad z = 0$$

by virtue of the relations

$$1 = \int_{-\infty}^{\infty} e^{i\alpha x}\delta(x)\,dx, \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\,d\alpha$$

expressing the (well-known) fact that $\delta(x)$ and 1 are Fourier transforms.

It may thus be seen, in summary, that the Green function $G(\mathbf{x};\boldsymbol{\xi},f,\epsilon)$ of the theory of flow about a body in regular waves (where the limit $\epsilon = +0$ is ultimately implied) satisfies the following equations:

$$\left.\begin{aligned} &\nabla^2 G = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) \quad \text{in} \quad z<0, && (4.3a)\\ &G_z - f(1+i\epsilon)^2 G = 0 \quad \text{on} \quad z = 0, && (4.3b)\\ &G = O(1/r) \quad \text{as} \quad r\to\infty, && (4.3c)\end{aligned}\right\} \text{ if } \zeta<0$$

$$\left.\begin{aligned} &\nabla^2 G = 0 \quad \text{in} \quad z<0, && (4.4a)\\ &G_z - f(1+i\epsilon)^2 G = -\delta(x-\xi)\delta(y-\eta) \quad \text{on} \quad z = 0, && (4.4b)\\ &G = O(1/r) \quad \text{as} \quad r\to\infty. && (4.4c)\end{aligned}\right\} \text{ if } \zeta = 0$$

As was noted previously, the Green function only depends on the three space variables $(x-\xi)^2 + (y-\eta)^2$, $(z-\zeta)^2$, and $(z+\zeta)$, so that this function is invariant under the substitution $\mathbf{x} \leftrightarrow \boldsymbol{\xi}$. Physically, the velocity potential Re $G(\mathbf{x};\boldsymbol{\xi},f,\epsilon)\exp(\epsilon-i)t$ of the flow created at point $\mathbf{x}(x,y,z\leqslant 0)$ by an outflow of strength Re $\exp(\epsilon-i)t$ at point $\boldsymbol{\xi}(\xi,\eta,\zeta\leqslant 0)$, stemming from a submerged source if $\zeta<0$ or a free-surface flux if $\zeta=0$, is identical to the potential Re $G(\boldsymbol{\xi};\mathbf{x},f,\epsilon)\exp(\epsilon-i)t$ of the flow created at point $\boldsymbol{\xi}$ by an outflow Re $\exp(\epsilon-i)$t at point $\mathbf{x}$, stemming from a source if $z<0$ or a free-surface flux if $z=0$. It then follows that equations (4.3) and (4.4) are also satisfied by the function $G(\boldsymbol{\xi};\mathbf{x},f,\epsilon)$. These equations are important for the formulation of an integral equation for determining the velocity potential of flow about a body in regular waves, as is shown in [19] where a new integral equation indeed is obtained by using both equations (4.3) and (4.4).

5. Single-integral representations

It is convenient to introduce the notation $h \equiv f\rho \equiv f(x'^2+y'^2)^{1/2}$, $v \equiv fz'$, and $d \equiv (h^2+v^2)^{1/2} \equiv fr'$, so that we have

$$\begin{aligned} h &\equiv f[(x-\xi)^2+(y-\eta)^2]^{1/2} \equiv \omega^2[(X-X_s)^2+(Y-Y_s)^2]^{1/2}/g,\\ v &\equiv f(z+\zeta) \equiv \omega^2(Z+Z_s)/g,\\ d &\equiv f[(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2]^{1/2}\\ &\equiv \omega^2[(X-X_s)^2+(Y-Y_s)^2+(Z+Z_s)^2]^{1/2}/g,\end{aligned}$$

where (X, Y, Z) and (X_s, Y_s, Z_s) are the dimensional coordinates of the field point and of the singularity, respectively, in the Green function, and equation (2.5a) was used. It may thus be seen that d represents the adimensional distance, in terms of $L = g/\omega^2$ as reference length, between the field point and the mirror image of the singularity with respect to the free-surface plane $z = 0$, while h is the horizontal distance (similarly adimensional) between these two points, and v is the negative of the vertical distance.

We restrict our attention to the limiting case $\epsilon = +0$ corresponding to purely oscillatory flow. By performing the change of variable $k = f\lambda$ in the inner integral in expression (3.10b), we can express the Green function $G(\mathbf{x}; \boldsymbol{\xi}, f) \equiv G(\mathbf{x}; \boldsymbol{\xi}, f, \epsilon = +0)$ in the form

$$4\pi G(\mathbf{x}; \boldsymbol{\xi}, f)/f = -1/fr + g(h, v), \tag{5.1}$$

where the function $g(h, v)$ is defined by the double integral

$$g(h, v) = -\frac{1}{d} - \frac{2}{\pi} \int_0^{\pi} d\theta \int_0^{\infty} d\lambda \frac{e^{(v - ih\cos\theta)\lambda}}{\lambda - (1 + i0)}. \tag{5.1a}$$

By performing the changes of variables $\alpha = f\mu$ and $\beta = f\nu$ in expression (3.10a), we may obtain the alternative double-integral representation

$$g(h, v) = \frac{1}{d} - \frac{2}{\pi} \int_0^{\infty} d\nu \int_{-\infty}^{\infty} d\mu \frac{e^{v(\mu^2 + \nu^2)^{1/2} - ih\mu}}{(\mu^2 + \nu^2)^{1/2} - (1 + i0)}. \tag{5.1b}$$

5.1 Haskind's integral representation

We start by expressing the double integral (5.1a) in the form

$$g(h, v) = -1/d - (2/\pi) \int_0^{\pi} I(\theta; h, v)\, d\theta, \tag{5.2}$$

where $I(\theta; h, v)$ is the inner integral given by

$$I(\theta; h, v) = \int_0^{\infty} \frac{e^{(v - ih\cos\theta)\lambda}}{\lambda - (1 + i0)} d\lambda.$$

By considering the contours of integration in the complex plane $\lambda \equiv \lambda_r + i\lambda_i$ shown in Figure 1 – where the lower and upper contours are selected for $0 < \theta < \pi/2$ and $\pi/2 < \theta < \pi$, respectively – we can express the integral I in the forms

$$I = \int_0^{\infty} \frac{e^{-(h\cos\theta + iv)t}}{t - i} dt \quad \text{for} \quad 0 < \theta < \pi/2,$$

$$I = \int_0^{\infty} \frac{e^{(h\cos\theta + iv)t}}{t + i} dt + 2\pi i e^{v - ih\cos\theta} \quad \text{for} \quad \pi/2 < \theta < \pi.$$

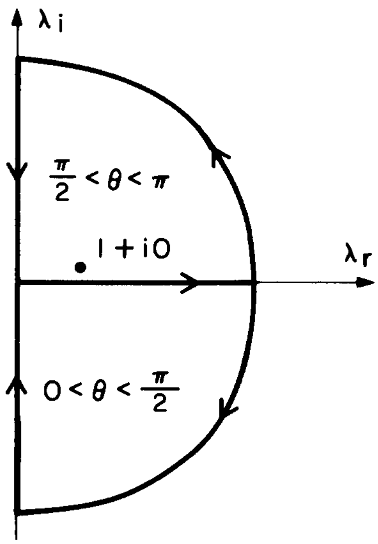

Figure 1. Contours of integration in the complex plane $\lambda \equiv \lambda_r + i\lambda_i$.

Use of the above expressions for I into equation (5.2) then readily yields

$$g(h,v) = -\frac{1}{d} - \frac{2}{\pi}\int_0^{\pi/2} d\theta \int_0^\infty \frac{e^{-(h\cos\theta + iv)t}}{t-i}\,dt - \frac{2}{\pi}\int_{\pi/2}^{\pi} d\theta \int_0^\infty \frac{e^{(h\cos\theta+iv)t}}{t+i}\,dt$$
$$-4ie^{v}\int_{\pi/2}^{\pi} e^{-ih\cos\theta}\,d\theta.$$

After performing the change of variable $\theta = \pi - \psi$ in the last two integrals, we may regroup the first two integrals and express the function $g(h, v)$ in the form

$$g(h,v) = W(h,v) + N(h,v), \tag{5.3}$$

where the functions $W(h,v)$ and $N(h,v)$ are defined as

$$W(h,v) = -4ie^{v}\int_0^{\pi/2} e^{ih\cos\theta}\,d\theta, \quad N(h,v) = -1/d - (2/\pi)\int_0^{\pi/2} J(\theta,h,v)\,d\theta, \tag{5.4}$$

with $J(\theta; h, v)$ given by

$$J = \int_0^\infty \frac{e^{-(h\cos\theta+iv)t}}{t-i}\,dt + \int_0^\infty \frac{e^{-(h\cos\theta-iv)t}}{t+i}\,dt. \tag{5.4a}$$

The integral $W(h, v)$ may be expressed in terms of 'standard functions', as may be seen for instance from equations (9.1.18) and (12.1.7) in Abramowitz and Stegun [24, pp. 360, 496]. Specifically, we have

$$W(h, v) = 2\pi \exp(v)[\tilde{H}_0(h) - iJ_0(h)], \tag{5.5}$$

where $\tilde{H}_0$ and J_0 are the usual Struve and Bessel functions, respectively.

By performing the changes of variables $\tau = (h \cos\theta + iv)t$ and $\tau = (h\cos\theta - iv)t$ in the first and second integrals, respectively, in expression (5.4a) for J, we may obtain

$$J = \int_0^\infty \frac{e^{-\tau} d\tau}{\tau + v - ih\cos\theta} + \int_0^\infty \frac{e^{-\tau} d\tau}{\tau + v + ih\cos\theta}. \tag{5.6}$$

By grouping these two integrals, we can obtain

$$J = 2\int_0^\infty \frac{e^{-\tau}(\tau + v)\, d\tau}{(\tau + v)^2 + h^2\cos^2\theta}.$$

Use of this expression for J in equation (5.4) then yields

$$N(h, v) = -\frac{1}{d} - \frac{4}{\pi}\int_0^\infty d\tau e^{-\tau}(\tau + v)\int_0^{\pi/2} \frac{d\theta}{(\tau + v)^2 + h^2\cos^2\theta},$$

where an interchange in the order of integration with respect to τ and θ was performed. The inner integral (with respect to θ) can be evaluated (in the usual manner by transforming this integral into an integral around the unit circle $|z| = 1$ in the complex plane $z = e^{i\theta}$). We may then obtain

$$N(h, v) = -\frac{1}{d} - 2\int_0^\infty \frac{e^{-\tau}\,\mathrm{sgn}\,(\tau + v)\, d\tau}{[(\tau + v)^2 + h^2]^{1/2}} = -\frac{1}{d} - 2e^v\int_v^\infty \frac{e^{-t}\,\mathrm{sgn}\, t\, dt}{(t^2 + h^2)^{1/2}},$$

where the change of variable $t = \tau + v$ was performed. The last expression may readily be written in the following alternative forms:

$$N(h, v) = -\frac{1}{d} - 2e^v\int_0^\infty e^{-t}(t^2 + h^2)^{-1/2} dt + 2e^v\int_v^0 e^{-t}(t^2 + h^2)^{-1/2} dt,$$

$$N(h, v) = -\frac{1}{d} - 4e^v\int_0^\infty e^{-t}(t^2 + h^2)^{-1/2} dt + 2e^v\int_v^\infty e^{-t}(t^2 + h^2)^{-1/2} dt.$$

The first integrals in the above alternative expressions can be expressed in terms of 'standard functions' as may be seen for instance from equation (12.1.8) [24, p. 496]. We may then obtain

$$N(h, v) = -\frac{1}{d} + \pi e^{v}[Y_0(h) - \tilde{H}_0(h)] + 2\int_0^{-v} e^{t+v}(t^2 + h^2)^{-1/2} dt, \quad (5.7a)$$

$$N(h, v) = -\frac{1}{d} + 2\pi e^{v}[Y_0(h) - \tilde{H}_0(h)] + 2\int_{-\infty}^{-v} e^{t+v}(t^2 + h^2)^{-1/2} dt, \quad (5.7b)$$

where Y_0 and $\tilde{H}_0$ are the usual Bessel and Struve functions, respectively, as they are defined in [24] for instance.

By using the above alternative expressions for $N(h, v)$ and expression (5.5) for $W(h, v)$ in equation (5.3), we may finally obtain

$$g(h, v) = \pi e^{v}[Y_0(h) + \tilde{H}_0(h) - 2iJ_0(h)] - \frac{1}{d} + 2\int_0^{-v} e^{t+v}(t^2 + h^2)^{-1/2} dt, \quad (5.8a)$$

$$g(h, v) = 2\pi e^{v}[Y_0(h) - iJ_0(h)] - \frac{1}{d} + 2\int_{-\infty}^{-v} e^{t+v}(t^2 + h^2)^{-1/2} dt. \quad (5.8b)$$

Expression (5.8b) is identical to the expression obtained by Haskind [7] and given in Wehausen and Laitone [12, p.477] equation (13.17′). The modified Haskind expression (5.8a) was used by Kim [14] and Yeung [15], and was also recently rederived by Hearn [16]. For purposes of numerical evaluation, a convenient alternative form of the integral in expression (5.8a) is obtained by performing the change of variable $\tau = -(t + v)/d$. This yields

$$g(h, v) = \pi e^{v}[Y_0(h) + \tilde{H}_0(h) - 2iJ_0(h)] - \frac{1}{d} + 2\int_0^{\alpha} e^{-d\tau}(1 - 2\alpha\tau + \tau^2)^{-1/2} d\tau, \quad (5.8c)$$

where we have $\alpha \equiv -v/d$ by definition and $0 \leqslant \alpha < 1$. This modified Haskind integral representation is very well suited for evaluating the Green function for small values of α. Indeed, for $v = 0$, the integral in formula (5.8c) vanishes, and we have the particularly simple expression

$$g(h, v = 0) = \pi[Y_0(h) + \tilde{H}_0(h) - 2iJ_0(h)] - 1/h. \quad (5.9)$$

However, Haskind's integral representation is clearly not well suited for evaluating the Green function for values of α close to 1. As a matter of fact, expressions (5.8) are not defined for $\alpha = 1$, i.e. for $h = 0$ and $v < 0$, so that these expressions can only be used for $v \leqslant 0$ and $h > 0$. A complementary single-integral representation that is well suited for evaluating the Green function for $v \leqslant 0$ and small values of $h \geqslant 0$ will now be derived.

5.2 The near-field integral representation

By performing the changes of variables $t = \tau + v - ih\cos\theta$ and $t = \tau + v + ih\cos\theta$ in the first and second integrals, respectively, in equation (5.6) we may obtain

$$J = \exp(v - ih\cos\theta)E_1(v - ih\cos\theta) + \exp(v + ih\cos\theta)E_1(v + ih\cos\theta),$$

where E_1 is the usual exponential integral defined here as in equation (5.1.1) [24, p.228]. By using the symmetry relation $E_1(\bar{Z}) = \overline{E_1(Z)}$, we may then obtain $J = 2 \text{ Re} \exp(v + ih \cos\theta) E_1(v + ih \cos\theta)$. Use of this expression for J in equation (5.4) finally yields

$$N(h,v) = -1/d - (4/\pi) \int_0^{\pi/2} \text{Re } e^Z E_1(Z) d\theta, \tag{5.10}$$

where Z is the complex function defined as $Z = v + ih \cos\theta$.

By using formulas (5.5) and (5.10) in formula (5.3), we then have

$$g(h,v) = 2\pi e^v [\tilde{H}_0(h) - iJ_0(h)] - \frac{1}{d} - \frac{4}{\pi} \int_0^{\pi/2} \text{Re } e^Z E_1(Z) d\theta; \quad Z \equiv v + ih \cos\theta. \tag{5.11}$$

For the sake of easy reference (and for reasons which will become clear further on), the expression for the Green function defined by formulas (5.1) and (5.11) is referred to as the 'near-field integral representation' of the Green function. This expression has also been obtained, independently and in a different manner, by Guevel and Daubisse [17] and Martin [18]. The near-field integral representation (5.11) takes a particularly simple form for $h = 0$, namely

$$g(h = 0, v) = 1/v - 2 \exp(v) [\text{Re } E_1(v + i0) + i\pi]. \tag{5.12}$$

The main interest of the integral representation (5.11), by comparison with the alternative integral representation (5.8c), resides in that this expression can be used to obtain an ascending series useful in the neighborhood of the origin $h = 0 = v$, i.e. for small values of h and $-v$. This ascending series will be given in Section 7.

5.3 The far-field integral representation

We now start from the double-integral representation (5.1b), which we write in the form

$$g(h,v) = 1/d - (2/\pi) \int_0^\infty I(\nu; h, v) d\nu, \tag{5.13}$$

where $I(\nu; h, v)$ is the inner integral defined as

$$I(\nu; h, v) = \int_{-\infty}^{\infty} \frac{e^{v(\mu^2 + \nu^2)^{1/2} - ih\mu}}{(\mu^2 + \nu^2)^{1/2} - (1 + i0)} d\mu.$$

By multiplying the numerator and denominator of the integrand of the inner integral I by the expression $(\mu^2 + \nu^2)^{1/2} + (1 + i0)$ and by rearranging the denominator, we may express this integral in the form

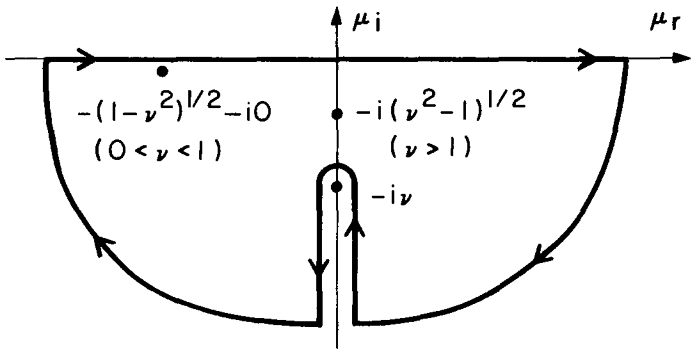

Figure 2. Contour of integration in the complex plane $\mu \equiv \mu_r + i\mu_i$.

$$I = \int_{-\infty}^{\infty} \frac{e^{v(\mu^2+\nu^2)^{1/2} - ih\mu}[(\mu^2+\nu^2)^{1/2} + (1+i0)]}{[\mu + (1-\nu^2+i0)^{1/2}][\mu-(1-\nu^2+i0)^{1/2}]} d\mu.$$

The poles $\pm(1-\nu^2+i0)^{1/2}$ of the integrand of the above integral are given by $\pm[(1-\nu^2)^{1/2}+i0]$ if $0<\nu<1$ and $\pm[0+i(\nu^2-1)^{1/2}]$ if $1<\nu<\infty$. By considering the contour of integration in the complex plane $\mu \equiv \mu_r + i\mu_i$ shown in Figure 2, and noting that we have $(\mu^2+\nu^2)^{1/2} = \mp i(\mu_i^2-\nu^2)^{1/2}$ for $\mu = \pm 0 + i\mu_i$ on the two sides of the cut defined by $\mu_r = 0$ and $-\infty < \mu_i < -\nu$, we can express the integral I in the form

$$I = \int_{-\infty}^{-\nu} \frac{e^{h\mu_i + iv(\mu_i^2-\nu^2)^{1/2}}}{i(\mu_i^2-\nu^2)^{1/2}-1} id\mu_i + \int_{-\nu}^{-\infty} \frac{e^{h\mu_i - iv(\mu_i^2-\nu^2)^{1/2}}}{-i(\mu_i^2-\nu^2)^{1/2}-1} id\mu_i - 2\pi iR. \tag{5.14}$$

In this expression R is the residue at the pole $-(1-\nu^2)^{1/2}$ if $0<\nu<1$, or at $-i(\nu^2-1)^{1/2}$ if $1<\nu<\infty$, so that we have

$$R(\nu;h,v) = \begin{cases} -\exp[v + ih(1-\nu^2)^{1/2}]/(1-\nu^2)^{1/2} & \text{if} \quad 0<\nu<1 \\ i\exp[v-h(\nu^2-1)^{1/2}]/(\nu^2-1)^{1/2} & \text{if} \quad 1<\nu<\infty \end{cases}, \tag{5.15}$$

as may easily be found. By performing the changes of variables $t = (\mu_i^2-\nu^2)^{1/2}$ and $t = -(\mu_i^2-\nu^2)^{1/2}$ in the first and second integrals, respectively, in expression (5.14), we can finally obtain

$$I(\nu;h,v) = \int_{-\infty}^{\infty} \frac{e^{-h(t^2+\nu^2)^{1/2} + ivt}}{(t+i)(t^2+\nu^2)^{1/2}} tdt - 2\pi iR(\nu;h,v), \tag{5.16}$$

where R is given by equations (5.15).

By using equations (5.16) and (5.15) in formula (5.13), we may then express the function $g(h,v)$ in the form

$$g(h,v) = W'(h,v) + N'(h,v), \tag{5.17}$$

where the functions W' and N' are given by

$$W'(h,v) = -4ie^{v}\int_0^1 \frac{e^{ih(1-\nu^2)^{1/2}}}{(1-\nu^2)^{1/2}}\,d\nu - 4e^{v}\int_1^\infty \frac{e^{-h(\nu^2-1)^{1/2}}}{(\nu^2-1)^{1/2}}\,d\nu, \tag{5.17a}$$

$$N'(h,v) = \frac{1}{d}-\frac{2}{\pi}\int_0^\infty d\nu\int_{-\infty}^\infty d\mu\,\frac{\mu e^{-h(\mu^2+\nu^2)^{1/2}+iv\mu}}{(\mu+i)(\mu^2+\nu^2)^{1/2}}. \tag{5.17b}$$

The two integrals in equation (5.17a) can be expressed in terms of usual Bessel functions, as may be seen by performing the changes of variables $\nu=\sin\theta$ and $\nu=(1+t^2)^{1/2}$ in the first and second integrals, respectively, and by using equation (9.1.18) [24, p.360] and equations (12.1.7) and (12.1.8) [24, p.496]. Specifically, we may obtain

$$W'(h,v) = 2\pi e^{v}[Y_0(h)-iJ_0(h)] \equiv -2\pi i e^{v}H_0^{(1)}(h). \tag{5.18}$$

By performing the changes of variables $\mu=\lambda\cos\theta$ and $\nu=\lambda\sin\theta$ in the double integral (5.17b), we may express the function $N'(h,v)$ in the form

$$N'(h,v) = 1/d-(2/\pi)\int_0^\pi J(\theta;h,v)\,d\theta, \tag{5.19}$$

where the inner integral J is given by

$$J(\theta;h,v) = \int_0^\infty \frac{e^{-(h-iv\cos\theta)\lambda}}{\lambda+i\sec\theta}\,\lambda\,d\lambda$$

This integral may be expressed as

$$J = \frac{1}{h-iv\cos\theta}-i\sec\theta\int_0^\infty \frac{e^{-(h-iv\cos\theta)\lambda}}{\lambda+i\sec\theta}\,d\lambda.$$

By performing the change of variable $\tau=\lambda+i\sec\theta$, we may then obtain

$$J = \frac{i\sec\theta}{Z}-i\sec\theta e^{Z}\int_{i\sec\theta}^\infty e^{-(h-iv\cos\theta)\tau}\,\frac{d\tau}{\tau},$$

where Z is the complex function defined as $Z=v+ih\sec\theta$. The change of variable $t=(h-iv\cos\theta)\tau$ then yields

$$J = i\sec\theta\left(\frac{1}{Z}-e^{Z}\int_Z^\infty \frac{e^{-t}}{t}\,dt\right) = i\sec\theta\,[1/Z-e^{Z}E_1(Z)].$$

Substitution of this expression for the inner integral J into formula (5.19) then gives

$$N'(h, v) = -1/d - (4/\pi) \int_0^{\pi/2} \operatorname{Im} e^Z E_1(Z) \sec\theta d\theta, \tag{5.20}$$

as may be found after some transformations.

By substituting expressions (5.18) and (5.20) for W' and N', respectively, into formula (5.17), we can finally obtain

$$g(h, v) = 2\pi e^v [Y_0(h) - iJ_0(h)] - \frac{1}{d} - \frac{4}{\pi} \int_0^{\pi/2} \operatorname{Im} e^Z E_1(Z) \sec\theta d\theta; \; Z \equiv v + ih \sec\theta. \tag{5.21}$$

This expression is obviously not defined for $h = 0$, and is best suited for evaluating the function $g(h, v)$ for large values of h. The expression for the Green function defined by formulas (5.1) and (5.21) will thus be referred to as the 'far-field integral representation'. To the author's knowledge, this integral representation does not seem to have been given previously. There is a striking similarity in form between the far-field representation (5.21) and the near-field representation (5.11). These two integral representations indeed are complementary. In particular, the near-field representation (5.11) readily provides an asymptotic expansion valid for large values of d and small or moderate values of h/d, that is in a sector neighboring the vertical axis $h = 0$; while an asymptotic expansion valid for large values of d and small or moderate values of $-v/d$, that is in a sector neighboring the horizontal axis $v = 0$, can readily be obtained from the far-field representation (5.21). These complementary asymptotic expansions are given in the following section. Comparison of expressions (5.8b) and (5.21) show that these expressions are equivalent, and that in fact we must have

$$-\frac{2}{\pi} \int_0^{\pi/2} \operatorname{Im} e^{v + ih\sec\theta} E_1(v + ih\sec\theta) \sec\theta d\theta \equiv \int_0^{\infty} e^{-d\tau} \left(1 + 2\frac{v}{d}\tau + \tau^2\right)^{-1/2} d\tau,$$

as may be obtained by performing the change of variable $\tau = -(t + v)/d$ in the integral in the Haskind expression (5.8b)

6. Asymptotic expansions

Let us first consider the far-field integral representation defined by expression (5.21), or by the equivalent equations (5.17), (5.18), and (5.20). We define the integral $I_1(h, v)$ as

$$I_1(h, v) = \operatorname{Im}(-2/\pi) \int_0^{\pi/2} (1/Z) \sec\theta d\theta; \qquad Z \equiv v + ih \sec\theta. \tag{6.1}$$

We have

$$I_1(h, v) = 1/d, \tag{6.1a}$$

as may easily be verified, and indeed was already used in the derivation of expression (5.20). By using equations (6.1 and 6.1a) in equation (5.20), we may express the function $N'(h, v)$ in the form

$$N'(h,v) = 1/d - (4/\pi)\int_0^{\pi/2} \mathrm{Im}\,[e^Z E_1(Z) - 1/Z]\sec\theta d\theta; \quad Z \equiv v + ih\sec\theta. \quad (6.2)$$

By using the well-known asymptotic expansion

$$\exp(Z)E_1(Z) - 1/Z \sim \sum_{n\geq 1} (-1)^n n!/Z^{n+1} \quad \text{as} \quad |Z| \to \infty, \quad \text{with} \quad |\mathrm{Arg}\, Z| < \pi, \quad (6.3)$$

in expression (6.2), we may obtain the asymptotic expansion

$$N'(h,v) \sim 1/d + 2\sum_{n\geq 1} (-1)^n n! I_{n+1}(h,v) \quad \text{as} \quad d \to \infty, \quad \text{with} \quad h > 0, \quad (6.4)$$

where $I_{n+1}(h, v)$ is the integral defined by

$$I_{n+1}(h,v) = \mathrm{Im}\,(-2/\pi)\int_0^{\pi/2} (1/Z^{n+1})\sec\theta d\theta; \qquad Z \equiv v + ih\sec\theta.$$

It may be seen that we have the relation

$$I_{n+1}(h,v) = (-1/n)\,\partial I_n(h,v)/\partial v, \quad (6.5)$$

from which we may obtain

$$(-1)^n n! I_{n+1}(h,v) = \partial^n I_1(h,v)/\partial v^n = \partial^n(1/d)/\partial v^n, \qquad (n \geq 1), \quad (6.5a)$$

where equation (6.1a) was used. The asymptotic expansion (6.4) then becomes

$$N'(h,v) \sim 1/d + 2\sum_{n\geq 1} \partial^n(1/d)/\partial v^n \quad \text{as} \quad d \to \infty, \quad \text{with} \quad h > 0. \quad (6.6)$$

It may be verified that we have

$$\partial^n(1/d)/\partial v^n = P_n(\alpha)/d^{n+1}, \quad (6.7)$$

where $\alpha \equiv -v/d$, and $P_n(\alpha)$ is a polynomial of degree n in α.

By using equation (5.18) and equations (6.6) and (6.7) in equation (5.17), we may finally obtain

$$g(h,v) \sim 2\pi e^{v}[Y_0(h) - iJ_0(h)] + 1/d + 2\sum_{n\geq 1} P_n(\alpha)/d^{n+1} \quad \text{as} \quad d \to \infty, \tag{6.8}$$

with $0 \leqslant \alpha \equiv -v/d < 1$. The first few polynomials $P_n(\alpha)$ may be shown to be

$$\begin{aligned} P_1 &= \alpha, & P_2 &= -(1-3\alpha^2), \\ P_3 &= -3^2\alpha(1-\tfrac{5}{3}\alpha^2), & P_4 &= 3^2(1-10\alpha^2+\tfrac{35}{3}\alpha^4), \\ P_5 &= 3^3\cdot 5^2\alpha(1-\tfrac{14}{3}\alpha^2+\tfrac{21}{5}\alpha^4), & P_6 &= -3^2\cdot 5^2(1-21\alpha^2+63\alpha^4-\tfrac{231}{5}\alpha^6). \end{aligned} \tag{6.8a}$$

We now consider the near-field integral representation defined by expression (5.11), or by the equivalent formulas (5.3), (5.5), and (5.10). We define the integral

$$I_1(h,v) = \mathrm{Re}\,(-2/\pi)\int_0^{\pi/2}(1/Z)\,d\theta; \qquad Z \equiv v + ih\cos\theta. \tag{6.9}$$

We have

$$I_1(h,v) = 1/d, \tag{6.9a}$$

as may be verified. By using equations (6.9) and (6.9a) in equation (5.10), we may express the function $N(h,v)$ in the form

$$N(h,v) = 1/d - (4/\pi)\int_0^{\pi/2}\mathrm{Re}\,[e^{Z}E_1(Z) - 1/Z]\,d\theta; \qquad Z \equiv v + ih\cos\theta. \tag{6.10}$$

By using the asymptotic expansion (6.3) in expression (6.10), we may obtain the asymptotic expansion

$$N(h,v) \sim 1/d + 2\sum_{n\geq 1}(-1)^n n!\,I_{n+1}(h,v) \quad \text{as} \quad d\to\infty, \quad \text{with} \quad v<0, \tag{6.11}$$

where $I_{n+1}(h,v)$ is the integral

$$I_{n+1}(h,v) = \mathrm{Re}\,(-2/\pi)\int_0^{\pi/2}(1/Z^{n+1})\,d\theta; \qquad Z \equiv v + ih\cos\theta.$$

Equations (6.5) and (6.5a) may readily be verified to hold, so that the asymptotic expansion (6.11) becomes

$$N(h,v) \sim 1/d + 2\sum_{n\geq 1}\partial^n(1/d)/\partial v^n \quad \text{as} \quad d\to\infty, \quad \text{with} \quad v<0. \tag{6.12}$$

The functions $N'(h, v)$ and $N(h, v)$ defined by equations (6.2) and (6.10) thus happen to have the same asymptotic expansion as $d \to \infty$, although these expansions are not valid in the same regions of the (h, v) plane. Specifically, the asymptotic expansions (6.6) and (6.12) are not valid in the neighborhoods of the vertical axis $h = 0$ and of the horizontal axis $v = 0$, respectively. It may be convenient to express the polynomials $P_n(\alpha)$ in equation (6.7) as functions of $\beta \equiv h/d$ [so that we have $\alpha \equiv (1 - \beta^2)^{1/2}$]. The polynomials $P_n(\alpha)$ can then be expressed in the form

$$P_n(\alpha) = n! Q_n(\beta), \tag{6.13}$$

where the functions $Q_n(\beta)$ verify $Q_n(0) = 1$, since we have $P_n(1) = n!$ as may be verified from equation (6.8a).

By using equation (5.5) and equations (6.12), (6.7), and (6.13) in equation (5.3), we may finally obtain

$$g(h, v) \sim 2\pi e^v [\widetilde{H}_0(h) - iJ_0(h)] + 1/d + 2 \sum_{n \geqslant 1} n! Q_n(\beta)/d^{n+1} \quad \text{as} \quad d \to \infty, \tag{6.14}$$

with $0 \leqslant \beta \equiv h/d < 1$. The first few functions $Q_n(\beta)$ are given by

$$\begin{aligned} Q_1 &= (1 - \beta^2)^{1/2}, & Q_2 &= 1 - \tfrac{3}{2}\beta^2, \\ Q_3 &= (1 - \beta^2)^{1/2}(1 - \tfrac{5}{2}\beta^2), & Q_4 &= 1 - 5\beta^2 + \tfrac{35}{8}\beta^4, \\ Q_5 &= (1 - \beta^2)^{1/2}(1 - 7\beta^2 + \tfrac{63}{8}\beta^4), & Q_6 &= 1 - \tfrac{21}{2}\beta^2 + \tfrac{189}{8}\beta^4 - \tfrac{231}{16}\beta^6. \end{aligned} \tag{6.14a}$$

The difference, $\delta g(h, v)$ say, between the values of the function $g(h, v)$ given by the asymptotic expansions (6.8) and (6.14) is given by

$$\delta g(h, v) = 2\pi e^v [Y_0(h) - \widetilde{H}_0(h)] \equiv 2\pi e^{-(1-\beta^2)^{1/2} d} [Y_0(\beta d) - \widetilde{H}_0(\beta d)]. \tag{6.15}$$

The function $\delta g(h, v)$ thus is exponentially small as $d \to \infty$, provided we have $0 < \beta < 1$. It may then be seen that the 'transition discontinuity' $\delta g(d, \beta = \beta_t)$ due to the use of the asymptotic expansions (6.14) and (6.8) for $0 \leqslant \beta < \beta_t$ and $\beta_t < \beta \leqslant 1$, respectively, is exponentially small, and thus is negligible – in an asymptotic sense – in comparison with the algebraic terms $1/d^n$ in the asymptotic expansions. An optimum transition between the asymptotic expansions (6.8) and (6.14) may be determined from the obvious requirement that the transition discontinuity $\delta g(d, \beta_t)$ is a minimum. This optimum transition then is given by the solution of the equation $\partial [\delta g(d, \beta)]/\partial \beta = 0$. By differentiating equation (6.15), we may then obtain the following equation for the 'transition curve' $v_t(h)$

$$-v_t = h [\widetilde{H}_0(h) - Y_0(h)] / [\widetilde{H}_1(h) - Y_1(h) - 2/\pi], \tag{6.16}$$

where equations (9.1.28) and (12.1.11) [24, pp. 361, 496] were used. In particular, equation (6.16) gives

$$-v_t \sim h^2(1 + 2/h^2 - 30/h^4 + \dots) \quad \text{as} \quad h \to \infty, \tag{6.16a}$$

as may be obtained by using equations (12.1.30) and (12.1.31) [24, p.497]. By substituting expression (6.16a) into equation (6.15) we may then obtain the following expression for the transition discontinuity $\delta g_t(h) \sim -4 \exp(-h^2)/h$ as $h \to \infty$. The discontinuity may be regarded as negligible in practice if it is sufficiently small in comparison with the main algebraic term, i.e. $1/d$, in the asymptotic expansions (6.8) and (6.14). We thus require that $4d \exp(-h^2)/h$ be smaller than the desired relative accuracy, ϵ say, which might be taken as $\epsilon = 0.01$ for practical applications. This then yields $4(1 + h^2)^{1/2} \exp(-h^2) < 0.01$ [since we have $d \equiv (h^2 + v^2)^{1/2} \sim h(1 + h^2)^{1/2}$ as $h \to \infty$ on the transition curve $-v \sim h^2$], from which we may obtain $h > 2.6$ and $d > 7.2$.

For sufficiently large values of d (say for d greater than about 7 according to the foregoing analysis), the function $g(h, v)$ can then be evaluated by means of the asymptotic expression

$$g(h,v) \sim W(h,v) + N(h,v) \quad \text{as} \quad d \to \infty. \tag{6.17}$$

The function $W(h, v)$ in expression (6.17) is given by

$$W(h,v) = \begin{cases} 2\pi \exp(v)[Y_0(h) - iJ_0(h)] & \text{for} \quad v_t(h) \leqslant v \leqslant 0 \\ 2\pi \exp(v)[\tilde{H}_0(h) - iJ_0(h)] & \qquad\; -\infty < v < v_t(h) \end{cases}, \tag{6.17a}$$

where the transition curve $v_t(h)$ is defined by equation (6.16). The function $N(h, v)$ in expression (6.17) can be evaluated by means of either one of the equivalent asymptotic expansions

$$N(h,v) \sim 1/d + 2 \sum_{n \geqslant 1} P_n(\alpha)/d^{n+1} \equiv 1/d + 2 \sum_{n \geqslant 1} n! Q_n(\beta)/d^{n+1} \quad \text{as} \quad d \to \infty, \tag{6.17b}$$

where $\alpha \equiv -v/d$ and $\beta \equiv h/d$, and the functions $P_n(\alpha)$ and $Q_n(\beta)$ are given by equations (6.8a) and (6.14a).

The error associated with the use of the asymptotic expansion (6.17b) is of the order of the term following the last term in the truncated series (i.e. the first discarded term in the series), as is well known. The requirement that the function $N(h, v)$ be evaluated with a relative accuracy ϵ (say with $\epsilon = 0.01$ in practice) may then be approximately expressed by the condition $2|P_n(\alpha)|/d^n < \epsilon$, which yields $d > [2|P_n(\alpha)|/\epsilon]^{1/n}$. The function $d_n(\alpha; \epsilon) \equiv [2|P_n(\alpha)|/\epsilon]^{1/n}$ may be evaluated, notably in the particular cases $\alpha = 0$ and $\alpha = 1$. In the particular case $\alpha = 1$ ($\beta = 0$), that is along the vertical axis $h = 0$, we have $P_n(1) = n!$, so that we may obtain $d_n(\alpha = 1, \epsilon) = (2n!/\epsilon)^{1/n}$. For $\epsilon = 0.01$, we may then obtain $d_1 = 200$, $d_2 = 20$, $d_3 = 10.63$, $d_4 = 8.32$, $d_5 = 7.52$, $d_6 = 7.24$, $d_7 = 7.21$, $d_8 = 7.30, \dots$. In the particular case $\alpha = 0 (\beta = 1)$, that is along the horizontal axis $v = 0$, we have $P_{2n-1}(0) = 0$ and $|P_{2n}(0)| = 1^2.3^2.5^2 \dots (2n-1)^2$, as may be seen from equations (6.8a). We may then obtain $d_{2n}(\alpha = 0, \epsilon) = [2.1^2. 3^2.5^2 \dots (2n-1)^2/\epsilon]^{1/2n}$. For $\epsilon = 0.01$, this yields $d_2 = 14.14$, $d_4 = 6.51$, $d_6 = 5.96$, $d_8 = 6.21 \dots$. The above results suggest that if a relative accuracy $\epsilon = 0.01$ is desired (as ought to be sufficient for most practical applications), it may not be advantageous to use more than the first five terms, i.e. $1 \leqslant n \leqslant 4$, in the asymptotic expansion (6.17b); furthermore, it appears that this 5-term asymptotic expansion could be used for d greater than about 8 if $\alpha = 1 (h = 0)$ and for d greater than about 6.5 if $\alpha = 0 (v = 0)$. These values of d fortunately happen to be about the same as the value $d = 7.2$ found previously from the requirement that the transition discontinuity in the value of the function $W(h, v)$ be negligible. On the basis of the foregoing analysis, it may thus be recommended that the 5-term asymptotic expansion (6.17b) be used in a region which may approximately (and tentatively) be defined by the equation $h^2/45 + v^2/65 \geqslant 1$; a more precise numerical determination of the domain of practical usefulness of the asymptotic expansion (6.17b) is, of course, possible.

7. Ascending series

In this section, an ascending series for the function $g(h, v)$ is obtained from the near-field representation given by formula (5.11), or by the equivalent equations (5.3), (5.5) and (5.10). Let the integrand $\exp(Z) E_1(Z)$ in equation (5.10) be expressed in the form

$$e^Z E_1(Z) = -e^Z (\ln Z + \gamma) + e^Z [E_1(Z) + \ln Z + \gamma]. \tag{7.1}$$

Furthermore, let the complex function $Z \equiv v + ih \cos\theta$ in the term $\ln Z$ be written in the form

$$Z = \frac{d-v}{2}\left(\frac{2v}{d-v} + i\frac{2h}{d-v}\cos\theta\right), \tag{7.2}$$

for reasons that will become clear further on. Also, let the parameter σ be defined as

$$\sigma \equiv h/(d-v). \tag{7.3}$$

It may be verified that we have $0 \leqslant \sigma \leqslant 1$, and $2v/(d-v) = \sigma^2 - 1$, so that equation (7.2) becomes

$$Z = \frac{d-v}{2}(\sigma^2 - 1 + i2\sigma\cos\theta).$$

Use of this expression for Z in equation (7.1) then yields

$$e^Z E_1(Z) = -e^{v+ih\cos\theta}\left[\ln\frac{d-v}{2} + \gamma + i\pi + \ln(1 - \sigma^2 - i2\sigma\cos\theta)\right]$$

$$+ e^Z [E_1(Z) + \ln Z + \gamma].$$

By using this expression for the integrand $\exp(Z)E_1(Z)$ in equation (5.10), we may express the function $N(h, v)$ in the form

$$N(h, v) = -\frac{1}{d} + 2e^v\left(\left[\ln\frac{d-v}{2} + \gamma\right]J_0(h) - \pi\widetilde{H}_0(h) + I\right) - 2J, \tag{7.4}$$

where equations (9.1.18) and (12.1.7) [24, pp. 360, 496] were used, and I and J are the integrals defined as

$$I = (2/\pi)\,\mathrm{Re}\int_0^{\pi/2} e^{ih\cos\theta}\ln\left[1 - \sigma^2 - i2\sigma\cos\theta\right]d\theta, \tag{7.5}$$

$$J = (2/\pi)\int_0^{\pi/2} \mathrm{Re}\, e^Z\left[E_1(Z) + \ln Z + \gamma\right]d\theta; \qquad Z \equiv v + ih\cos\theta. \tag{7.6}$$

Substitution of expressions (5.5) and (7.4) for the functions W and N in equation (5.3) finally yields the expression

$$g(h, v) = -1/d + 2e^v\left[\left(\ln\frac{d-v}{2} + \gamma - i\pi\right)J_0(h) + I\right] - 2J. \tag{7.7}$$

The ascending series for the above-defined integrals I and J are given below. The integral J is considered first.
We have

$$e^Z\left[E_1(Z) + \ln Z + \gamma\right] = \left[\sum_{m=0}^{\infty} Z^m/m!\right]\left[\sum_{k=1}^{\infty}(-1)^{k+1}Z^k/k\cdot k!\right],$$

as readily follows from the ascending series of the functions $\exp(Z)$ and $E_1(Z)$. The above product of series may be expressed in the form

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\frac{n!}{(n-k)!k!}\right]\frac{Z^n}{n!} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)\frac{Z^n}{n!},$$

where equation (0.155, 4) [25, p.4] was used. The well-known binomial expansion formula yields

$$Z^n \equiv (v + ih\cos\theta)^n = \sum_{k=0}^{n}\binom{n}{k} i^k h^k v^{n-k}\cos^k\theta.$$

We may then obtain

$$\mathrm{Re}\, Z^n = n!\sum_{k=0}^{n'}(-1)^k h^{2k} v^{n-2k}\cos^{2k}\theta/(2k)!(n-2k)!,$$

where n' is defined as $n' = n/2$ if n is even or as $n' = (n-1)/2$ if n is odd. We thus have

$$\operatorname{Re} e^{Z}[E_1(Z) + \ln Z + \gamma] = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \frac{1}{m}\right) \left[\sum_{k=0}^{n'} (-1)^k \frac{h^{2k} v^{n-2k}}{(2k)!(n-2k)!} \cos^{2k}\theta\right].$$

Substitution of this series for the integrand in the integral (7.6) and term by term integration finally yields

$$J = v + \sum_{n=2}^{\infty} \left(\sum_{m=1}^{n} \frac{1}{m}\right)\left(v^n/n! + \sum_{k=1}^{n'} (-1)^k \frac{1.3.5...(2k-1)}{2.4.6...(2k)} \frac{h^{2k} v^{n-2k}}{(2k)!(n-2k)!}\right);$$

$$n' = \begin{cases} n/2 \\ (n-1)/2 \end{cases}. \tag{7.8}$$

We now consider the integral I. By replacing the function $\exp(ih\cos\theta)$ in equation (7.5) by the ascending series $\sum_{n=0}^{\infty} i^n h^n \cos^n\theta/n!$, we may obtain

$$I = \sum_{n=0}^{\infty} (-1)^n \left[\frac{h^{2n}}{(2n)!} \operatorname{Re} I_{2n} - \frac{h^{2n+1}}{(2n+1)!} \operatorname{Im} I_{2n+1}\right], \tag{7.9}$$

where I_n is the integral defined by

$$I_n = (2/\pi) \int_0^{\pi/2} \ln[1 - \sigma^2 - i2\sigma\cos\theta] \cos^n\theta \, d\theta.$$

It may be verified that we have

$$\operatorname{Re} I_{2n} = I'_{2n} \quad \text{and} \quad \operatorname{Im} I_{2n+1} = -iI'_{2n+1}, \tag{7.10}$$

where I'_n is the integral given by

$$I'_n = (1/2\pi) \int_{-\pi}^{\pi} \ln[1 - \sigma^2 - i2\sigma\cos\theta] \cos^n\theta \, d\theta.$$

Use of equations (7.10) in equation (7.9) then yields

$$I = I'_0 + \sum_{n=1}^{\infty} (-1)^n \left[\frac{h^{2n}}{(2n)!} I'_{2n} - i \frac{h^{2n-1}}{(2n-1)!} I'_{2n-1}\right]. \tag{7.11}$$

The integral I'_n can be expressed as a contour integral around the unit circle $|z| = 1$ in the complex plane $z = \exp(i\theta)$. We thus have

$$i2^n I'_n = \frac{1}{2\pi} \int_{|z|=1} \ln\left[-\frac{i\sigma}{z}\left(z+\frac{i}{\sigma}\right)(z-i\sigma)\right]\left(z+\frac{1}{z}\right)^n \frac{dz}{z}. \tag{7.12}$$

By using the binomial theorem, and after some transformations, we may obtain

$$\frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} = \sum_{k=0}^{n-1}\binom{2n}{k}\left(z^{2n-2k-1}+\frac{1}{z^{2n-2k+1}}\right)+\binom{2n}{n}\frac{1}{z}, \tag{7.13a}$$

$$\frac{1}{z}\left(z+\frac{1}{z}\right)^{2n-1} = \sum_{k=0}^{n-1}\binom{2n-1}{k}\left(z^{2n-2k-2}+\frac{1}{z^{2n-2k}}\right). \tag{7.13b}$$

Use of equations (7.13a,b) in equation (7.12) then yields

$$i2^{2n} I'_{2n} = i\binom{2n}{n} I'_0 + \sum_{k=0}^{n-1}\binom{2n}{k}(I^+_{2n-2k-2}+I^-_{2n-2k+1}), \tag{7.14a}$$

$$i2^{2n-1} I'_{2n-1} = \sum_{k=0}^{n-1}\binom{2n-1}{k}(I^+_{2n-2k-2}+I^-_{2n-2k}) \tag{7.14b}$$

where I^+_m and I^-_m are the integrals defined by

$$I^+_m = \frac{1}{2\pi}\int_{|z|=1} \ln\left[-\frac{i\sigma}{z}\left(z+\frac{i}{\sigma}\right)(z-i\sigma)\right] z^m dz, \qquad m \geqslant 0, \tag{7.15a}$$

$$I^-_m = \frac{1}{2\pi}\int_{|z|=1} \ln\left[-\frac{i\sigma}{z}\left(z+\frac{i}{\sigma}\right)(z-i\sigma)\right] \frac{dz}{z^m}, \qquad m \geqslant 2. \tag{7.15b}$$

It may be verified that we have

$$I'_0 = 0. \tag{7.16}$$

Furthermore, we have

$$I^-_m = I^+_{m-2}, \tag{7.17}$$

as may easily be verified by performing the change of variable $z = 1/\zeta$ in the integral (7.15b). Use of equations (7.16) and (7.17) in equations (7.14a,b) then yields

$$i2^{2n-1} I'_{2n} = \sum_{k=0}^{n-1}\binom{2n}{k} I^+_{2n-2k-1}, \tag{7.18a}$$

$$i2^{2n-2}I'_{2n-1} = \sum_{k=0}^{n-1} \binom{2n-1}{k} I^{+}_{2n-2k-2}. \tag{7.18b}$$

We have $0 \leqslant \sigma \leqslant 1$, so that the function $\ln[-i\sigma(z + i/\sigma)]\, z^m$ is holomorphic in the region $|z| \leqslant 1$, and the integral (7.15a) becomes

$$I^{+}_{m} = \frac{1}{2\pi} \int_{|z|=1} \ln\left(\frac{z - i\sigma}{z}\right) z^m dz = \frac{1}{2\pi} \int_{|\zeta|=1} \ln(1 - i\sigma\zeta)\, \frac{d\zeta}{\zeta^{m+2}}, \tag{7.19}$$

where the change of variable $\zeta = 1/z$ was performed. We can then obtain

$$I^{+}_{m} = i^m \sigma^{m+1}/(m+1). \tag{7.20}$$

Use of expression (7.20) for the integral I^{+}_{m} in equations (7.18a,b) then gives

$$(-1)^n I'_{2n}/(2n)! = (-2/2^{2n}) \sum_{k=0}^{n-1} (-1)^k \sigma^{2n-2k}/(2n-2k)\,k!\,(2n-k)!, \tag{7.21a}$$

$$i(-1)^n I'_{2n-1}/(2n-1)! =$$

$$(-2/2^{2n-1}) \sum_{k=0}^{n-1} (-1)^k \sigma^{2n-1-2k}/(2n-1-2k)\,k!\,(2n-1-k)!. \tag{7.21b}$$

By substituting expression (7.16) and (7.21a,b) for the integrals I'_0, I'_{2n} and I'_{2n-1} in equation (7.11), we may finally obtain

$$I = -2 \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n'} \frac{(-1)^k}{k!(n-k)!} \frac{\sigma^{n-2k}}{n-2k} \right) \left(-\frac{h}{2}\right)^n ; \quad \sigma \equiv \frac{h}{d-v}, \quad n' = \begin{cases} n/2 - 1 \\ (n-1)/2 \end{cases}. \tag{7.22}$$

Equations (7.22), (7.8), and (7.7) – where the classical ascending series for the Bessel and Struve functions $J_0(h)$ and $\tilde{H}_0(h)$, namely

$$J_0(h) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{h^2}{4}\right)^n /(n!)^2, \tag{7.23a}$$

$$\frac{\pi}{2}\tilde{H}_0(h) = \sum_{n=0}^{\infty} (-1)^n h^{2n+1}/1^2 . 3^2 . 5^2 \ldots (2n+1)^2, \tag{7.23b}$$

may evidently be used – provide an ascending series useful for evaluating the function $g(h, v)$ for small and moderate values of d.

8. One-dimensional Taylor-series expansions

The near-field representation (5.11) may be used to obtain a Taylor-series expansion of the function $g(h, v)$ in the neighborhood of the vertical axis $h = 0$. Let $I(h, v)$ be the function defined as

$$I(h,v) = \mathrm{Re}\,(4/\pi) \int_0^{\pi/2} e^Z E_1(Z)\,d\theta; \qquad Z \equiv v + ih\cos\theta. \tag{8.1}$$

By expanding the function $I(h, v)$ in a Taylor series about the axis $h = 0$, we may obtain

$$I(h,v) = \sum_{n=0}^{\infty} h^n\,[\partial^n I(h,v)/\partial h^n]_{h=0}/n!. \tag{8.2}$$

Differentiation of both sides of equation (8.1) yields

$$\partial^n I(h,v)/\partial h^n = \mathrm{Re}\,(4/\pi) \int_0^{\pi/2} [d^n e^Z E_1(Z)/dZ^n]\, i^n \cos^n\theta\, d\theta. \tag{8.3}$$

From the definition of the exponential integral function $E_1(Z)$, given for instance by equation (5.1.1) [24, p. 228], one may show that

$$d^n e^Z E_1(Z)/dZ^n = e^Z E_1(Z) + \sum_{k=1}^{n} (k-1)!/(-Z)^k \quad \text{for} \quad n \geqslant 1. \tag{8.4}$$

By using equation (8.4) in equation (8.3), we may obtain

$$\left.\frac{\partial^{2n} I}{\partial h^{2n}}\right|_{h=0} = 2(-1)^n \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \left[e^v\,\mathrm{Re}\,E_1(v+i0) + \sum_{k=1}^{2n} \frac{(k-1)!}{(-v)^k}\right] \quad \text{for} \quad n \geqslant 1, \tag{8.5a}$$

while for $n = 0$ we have

$$I(h = 0, v) = 2e^v\,\mathrm{Re}\,E_1(v + i0). \tag{8.5b}$$

We have $\mathrm{Re}\, iE_1(v + i0) = -\mathrm{Im}\, E_1(v + i0) = \pi$, as may be found from the ascending series for the exponential integral given, for instance, by equation (5.1.11) [24, p.229]. It may then be seen that we have

$$\partial I/\partial h|_{h=0} = 4e^{v}, \tag{8.5c}$$

and

$$\left.\frac{\partial^{2n+1} I}{\partial h^{2n+1}}\right|_{h=0} = 4(-1)^n \frac{2.4.6\ldots(2n)}{3.5.7\ldots(2n+1)} e^{v} \quad \text{for} \quad n \geqslant 1. \tag{8.5d}$$

Use of equations (8.5a,b,c,d) in the series (8.2) then yields

$$I(h,v) = 2e^{v} \operatorname{Re} E_1(v+i0)\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \frac{h^{2n}}{(2n)!}\right] + 2J(h,v)$$

$$+ 4e^{v}\left[h + \sum_{n=1}^{\infty} (-1)^n \frac{2.4.6\ldots(2n)}{3.5.7\ldots(2n+1)} \frac{h^{2n+1}}{(2n+1)!}\right], \tag{8.6}$$

where the function $J(h, v)$ is defined by the series

$$J(h,v) = \sum_{n=1}^{\infty} (-1)^n \left[\sum_{k=1}^{2n} \frac{(k-1)!}{(-v)^k}\right] \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \frac{h^{2n}}{(2n)!}. \tag{8.6a}$$

The two series between brackets in equation (8.6) are the ascending series for the functions $J_0(h)$ and $(\pi/2)\tilde{H}_0(h)$, respectively, as may readily be verified from equations (7.23a,b), so that equation (8.6) becomes

$$I(h,v) = 2e^{v}[\operatorname{Re} E_1(v+i0) J_0(h) + \pi \tilde{H}_0(h)] + 2J(h,v). \tag{8.7}$$

The series (8.6a) for the function $J(h, v)$ may be expressed in the form

$$J(h,v) = \sum_{n=1}^{\infty} (-1)^n \left[\sum_{m=0}^{2n-1} \frac{(2n-1-m)!}{(-v)^{2n-m}}\right] \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \frac{h^{2n}}{(2n)!}. \tag{8.7a}$$

By substituting expression (8.7) for the integral (8.1) into formula (5.11), we may finally express the function $g(h, v)$ in the form

$$g(h,v) = -1/d - 2J_0(h) \exp(v)[\operatorname{Re} E_1(v+i0) + i\pi] - 2J(h,v). \tag{8.8}$$

The series (8.7a) for the function $J(h, v)$ can be written in the form

$$J(h,v) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)} \left(\frac{h}{-v}\right)^{2n} P_n(v), \tag{8.8a}$$

where $P_n(v)$ is the polynomial of degree $2n-1$ given by

$$P_n(v) = 1 + \sum_{m=1}^{2n-1} (-v)^m/(2n-1)(2n-2)(2n-3)\dots(2n-m). \tag{8.8b}$$

We have $J(h=0, v)=0$ and $J_0(h=0)=1$, so that expression (8.8) becomes identical to expression (5.12) in the limit $h=0$.

A complementary Taylor series of the function $g(h, v)$ in the neighborhood of the horizontal axis $v=0$ can be obtained from the Haskind integral representation (5.8a). Let $I(h, v)$ be the function defined as

$$I(h, v) = \int_0^{-v} e^{t+v}(t^2+h^2)^{-1/2}dt. \tag{8.9}$$

Expansion of this function in a Taylor series about the axis $v=0$ yields

$$I(h, v) = \sum_{n=1}^{\infty} v^n [\partial^n I(h, v)/\partial v^n]_{v=0}/n!, \tag{8.10}$$

where the fact that $I(h, v=0)=0$ was used. It can be verified that we have

$$\left.\frac{\partial^{2n+1}I}{\partial v^{2n+1}}\right|_{v=0} = \sum_{k=0}^{n} (-1)^{k+1} \frac{1^2.3^2.5^2\dots(2k-1)^2}{h^{2k+1}} = \left.\frac{\partial^{2n+2}I}{\partial v^{2n+2}}\right|_{v=0} \quad \text{for} \quad n \geqslant 0. \tag{8.11}$$

Use of equation (8.11) in equation (8.10) yields the series

$$I(h, v) = \sum_{n=0}^{\infty} \frac{v^{2n+1}}{(2n+1)!}\left(1+\frac{v}{2n+2}\right)\left[\sum_{k=0}^{n}(-1)^{k+1}\frac{1^2.3^2.5^2\dots(2k-1)^2}{h^{2k+1}}\right],$$

which may be written in the equivalent form

$$I(h, v) = \sum_{n=0}^{\infty} \frac{(-v)^{2n+1}}{(2n+1)!}\left(1+\frac{v}{2n+2}\right)\left[\sum_{m=0}^{n}(-1)^{n-m}\frac{1^2.3^2.5^2\dots(2n-1-2m)^2}{h^{2n+1-2m}}\right]. \tag{8.12}$$

By using equation (8.9) in equation (5.8), we may then express the function $g(h, v)$ in the form

$$g(h, v) = -1/d + \pi \exp(v)[Y_0(h) + \widetilde{H}_0(h) - 2iJ_0(h)] + 2I(h, v). \tag{8.13}$$

The series (8.12) for the function $I(h, v)$ may be written as

$$I(h,v) = \frac{-v}{h}\left(1+\frac{v}{2}\right)+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\,\frac{1.3.5...(2n-1)}{2.4.6...(2n)}\left(\frac{-v}{h}\right)^{2n+1}\left(1+\frac{v}{2n+2}\right)P_n(h), \qquad (8.13a)$$

where $P_n(h)$ is the polynomial of degree $2n$ given by

$$P_n(h) = 1+\sum_{m=1}^{n}(-1)^m h^{2m}/(2n-1)^2(2n-3)^2(2n-5)^2...(2n+1-2m)^2. \qquad (8.13b)$$

In the limit $v = 0$, we have $I = 0$, and expression (8.13) becomes identical to expression (5.9).

9. The gradient of the Green function

The vertical derivative, G_z, of the Green function $G(\mathbf{x};\boldsymbol{\xi},f)$ may be expressed directly in terms of the function $g(h, v)$, as will now be shown. Following an idea used by Eggers [22] for the similar problem of ship wave resistance, we express the Green function in the alternative forms

$$4\pi G(\mathbf{x};\boldsymbol{\xi},f) = -1/r+1/r'+H^+(\rho,z';f) = -1/r-1/r'+H^-(\rho,z';f), \qquad (9.1a,b)$$

where we have $\rho\equiv[(x-\xi)^2+(y-\eta)^2]^{1/2}$, $z'\equiv z+\zeta$, $r\equiv[\rho^2+(z-\zeta)^2]^{1/2}$, $r'\equiv(\rho^2+z'^2)^{1/2}$ and $f\equiv\omega^2L/g$, as was defined previously. Although the precise expressions for the functions $H^+(\rho,z';f)$ and $H^-(\rho,z';f)$ can evidently be readily obtained from the analysis in the previous sections, for instance by setting $\epsilon=+0$ in expressions (3.10a) and (3.10b), the precise forms of these functions are actually not required here. By using equations (9.1a, b), we may obtain

$$4\pi(G_z-fG) = (-1/r-1/r')_z+H_z^- - f(-1/r+1/r')-fH^+. \qquad (9.2)$$

The sea-surface condition (4.3b) shows that we have $G_z-fG=0$ on $z=0$ if $\zeta<0$. It may also readily be seen that $-1/r+1/r'=0$ on $z=0$ and $(1/r+1/r')_z=0$ on $z=0$ if $\zeta<0$. It then follows from equation (9.2) that we have $H_z^- - fH^+=0$ on $z=0$ if $\zeta<0$. This relation however must hold for $z\leqslant 0$, since the functions H^+ and H^- depend on $z+\zeta$. We thus have $H_z^- - fH^+\equiv 0$, as may also readily be verified from equations (3.9a) and (3.9b). Equation (9.2) then becomes

$$4\pi(G_z-fG) = -(1/r+1/r')_z+f(1/r-1/r'). \qquad (9.3)$$

By using expression (5.1) for the green function, that is

$$4\pi G(\mathbf{x};\boldsymbol{\xi},f)/f = -1/fr+g(h,v) \qquad (9.4)$$

in equation (9.3), we may finally obtain

$$4\pi G_z/f^2 \;=\; (z-\zeta)/f^2r^3 + v/d^3 - 1/d + g(h,v), \tag{9.5}$$

where $v \equiv fz'$, $h \equiv f\rho$, and $d \equiv fr'$, as was defined previously. Expression (9.5) for the vertical derivative G_z of the Green function was obtained previously by Martin [18], in a different manner. The horizontal derivatives G_x and G_y of the Green function G may readily be obtained by differentiating expression (9.4). Specifically, we may obtain

$$G_x \;=\; G_\rho(x-\xi)/\rho, \qquad G_y \;=\; G_\rho(y-\eta)/\rho, \tag{9.6a,b}$$

with $G_\rho \equiv \partial G/\partial\rho$ given by

$$4\pi G_\rho/f^2 \;=\; \rho/f^2r^3 + g_h(h,v). \tag{9.7}$$

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